

# Spinorially twisted Spin structures, I: curvature identities and eigenvalue estimates

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## Abstract

We define (higher rank) spinorially twisted spin structures and deduce various curvature identities as well as estimates for the eigenvalues of the corresponding twisted Dirac operators.

## 1 Introduction

The aim of this note is to introduce (higher rank) spinorially twisted Spin structures and prove various curvature formulas and eigenvalue estimates for the corresponding twisted Dirac operators. Such formulas and estimates are the higher rank analogues of those proved by Hitchin [4], Friedrich [1], Hijazi [3] and many others. We begin by noticing that a  $\text{Spin}^c$  structure on a Riemannian  $n$ -dimensional manifold  $M$  consists of the coupling of a (locally defined)  $\text{Spin}(n)$ -structure and an auxiliary (locally defined)  $U(1) = \text{Spin}(2)$  structure [1] and, similarly, a  $\text{Spin}^q$  structure on  $M$  consists of the coupling of a (locally defined)  $\text{Spin}(n)$ -structure and an auxiliary (locally defined)  $Sp(1) = \text{Spin}(3)$  structure [7]. Here, we consider analogous twistings with other  $\text{Spin}(r)$  groups,  $r \geq 2$ , in an attempt to develop spinorial techniques to study the geometry of manifolds which are neither Spin, nor  $\text{Spin}^c$ , nor  $\text{Spin}^q$ . Although the idea of “twisting” is a classical one, here we will take advantage of the spin geometry (and the Clifford algebra representation) carried by the spinorial twists.

A twisted Spin group is defined as

$$\text{Spin}^r(n) = \text{Spin}(n) \times_{\mathbb{Z}_2} \text{Spin}(r),$$

where  $n, r \in \mathbb{N}$ . Each factor has a standard spin representation  $\Delta_n$  and  $\Delta_r$  respectively, so that we can consider the representation of  $\text{Spin}^r(n)$

$$\Delta_n \otimes \Delta_r,$$

and more generally

$$\Delta_n \otimes \Delta_r^{\otimes m}$$

for odd  $m \in \mathbb{N}$ . These are the representations that we will use to associate twisted spinor vector bundles to an  $n$ -dimensional oriented Riemannian manifold  $M$  admitting a  $\text{Spin}^r(n)$  structure (cf. Definition 2.2), whose sections will be called spinor fields, or simply spinors. By choosing a connection on the induced/auxiliary  $SO(r)$  principal bundle on  $M$ , together with the Levi-Civita connection on  $TM$ , one can construct a connection on the twisted spinor bundles and talk about parallel and Killing spinors, as well as twisted Dirac operators and connection Laplacians.

As in the Spin,  $\text{Spin}^c$  and  $\text{Spin}^q$  cases, one can prove (spinorial) curvature identities (cf. (7), (8), (9)). Such identities render new formulas for the Ricci and the scalar curvatures of  $M$  in the presence of either a parallel spinor (cf. Theorem 3.1) or a Killing spinor (cf. Theorem 3.2), thanks to the introduction of local 2-forms associated to any spinor (cf. Definition 2.1). These formulas involve not only the curvature of the

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connection on the auxiliary bundle, but also the aforementioned 2-forms, whose appearance is made possible by the introduction of the spinorial twist. Clearly, the existence of parallel or Killing spinors on a manifold will impose restrictions on its geometry (holonomy), but this topic will be treated elsewhere [2].

We also present a twisted version of the Schrödinger-Lichnerowicz formula involving the twisted Dirac operator and the connection Laplacian, and use the Bochner technique to derive various eigenvalue estimates analogous to those contained in [4, 1, 3, 8]. We would like to point out that our estimates involve the tensor power  $m$  of the twisting bundle (see Corollary 3.1). On the other hand, Corollaries 3.2 and 3.4 give new spinor specific criteria to check whether a spinor is parallel, or whether a number can be a Killing constant in terms of the associated 2-forms of the spinor and the curvature of the auxiliary bundle.

The paper is organized as follows. In Section 2 we recall basic material [1] regarding Clifford algebras, Clifford multiplication, Spin groups, etc., and define the twisted Spin groups, twisted spin representations and the 2-forms associated to spinors. We also recall some material about connections on (twisted) spin bundles and define the twisted Dirac operator and connection Laplacian. In Section 3, we prove various spinorial curvature identities analogous to those in the  $\text{Spin}^c$  case (Subsection 3.1) and apply them in the cases of parallel and Killing spinors (Subsections 3.2 and 3.3). We develop in detail a particular example of parallel twisted spinor on any oriented Riemannian manifold (cf. Proposition 3.3), with the interesting feature that its associated 2-forms generate a copy of  $\mathfrak{so}(n)$  (cf. Proposition 3.4). Finally, we prove the twisted Schrödinger-Lichnerowicz formula (Theorem 3.4) and prove several corollaries (Subsection 3.5).

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## 2 Preliminaries

In this section, we recall basic material that can be consulted in [1] and define the various twisted objects that will be used throughout.

### 2.1 Clifford algebra, spin group and representation

Let  $Cl_n$  denote the  $2^n$ -dimensional real Clifford algebra generated by the orthonormal vectors  $e_1, e_2, \dots, e_n \in \mathbb{R}^n$  subject to the relations  $e_i e_j + e_j e_i = -2\delta_{ij}$ , and  $\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$  denote its complexification. Recall that

$$\mathbb{C}l_n \cong \begin{cases} \text{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k, \\ \text{End}(\mathbb{C}^{2^k}) \oplus \text{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k + 1, \end{cases}$$

and that the space of (complex) spinors is defined as

$$\Delta_n := \mathbb{C}^{2^k} = \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{k \text{ times}}.$$

The map

$$\kappa : \mathbb{C}l_n \longrightarrow \text{End}(\mathbb{C}^{2^k})$$

is defined to be either the aforementioned isomorphism for  $n$  even, or the isomorphism followed by the projection onto the first summand for  $n$  odd. In order to make  $\kappa$  explicit, consider the following matrices

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In terms of the generators  $e_1, \dots, e_n$  of the Clifford algebra,  $\kappa$  can be described explicitly as follows,

$$\begin{aligned} e_1 &\mapsto Id \otimes Id \otimes \dots \otimes Id \otimes Id \otimes g_1, \\ e_2 &\mapsto Id \otimes Id \otimes \dots \otimes Id \otimes Id \otimes g_2, \\ e_3 &\mapsto Id \otimes Id \otimes \dots \otimes Id \otimes g_1 \otimes T, \\ e_4 &\mapsto Id \otimes Id \otimes \dots \otimes Id \otimes g_2 \otimes T, \end{aligned}$$

$$\begin{aligned}
& \vdots \quad \dots \\
e_{2k-1} & \mapsto g_1 \otimes T \otimes \dots \otimes T \otimes T \otimes T, \\
e_{2k} & \mapsto g_2 \otimes T \otimes \dots \otimes T \otimes T \otimes T,
\end{aligned}$$

and, if  $n = 2k + 1$ ,

$$e_{2k+1} \mapsto i T \otimes T \otimes \dots \otimes T \otimes T \otimes T.$$

The vectors

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i) \quad \text{and} \quad u_{-1} = \frac{1}{\sqrt{2}}(1, i),$$

form a unitary basis of  $\mathbb{C}^2$  with respect to the standard Hermitian product. Thus,

$$\{u_{\varepsilon_1, \dots, \varepsilon_k} = u_{\varepsilon_1} \otimes \dots \otimes u_{\varepsilon_k} \mid \varepsilon_j = \pm 1, j = 1, \dots, k\},$$

is a unitary basis of  $\Delta_n = \mathbb{C}^{2^k}$  with respect to the naturally induced Hermitian product. We will denote inner and Hermitian products (as well as Riemannian and Hermitian metrics) by the same symbol  $\langle \cdot, \cdot \rangle$  trusting that the context will make clear which product is being used.

By means of  $\kappa$  we have Clifford multiplication

$$\begin{aligned}
\mu_n : \mathbb{R}^n \otimes \Delta_n & \longrightarrow \Delta_n \\
x \otimes \phi & \mapsto \mu_n(x \otimes \phi) = x \cdot \phi := \kappa(x)(\phi).
\end{aligned}$$

A quaternionic structure  $\alpha$  on  $\mathbb{C}^2$  is given by

$$\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix},$$

and a real structure  $\beta$  on  $\mathbb{C}^2$  is given by

$$\beta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}.$$

Following [1, p. 31], the real and quaternionic structures  $\gamma_n$  on  $\Delta_n = (\mathbb{C}^2)^{\otimes [n/2]}$  are built as follows

$$\begin{aligned}
\gamma_n &= (\alpha \otimes \beta)^{\otimes 2k} & \text{if } n = 8k, 8k + 1 & \quad (\text{real}), \\
\gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k} & \text{if } n = 8k + 2, 8k + 3 & \quad (\text{quaternionic}), \\
\gamma_n &= (\alpha \otimes \beta)^{\otimes 2k+1} & \text{if } n = 8k + 4, 8k + 5 & \quad (\text{quaternionic}), \\
\gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k+1} & \text{if } n = 8k + 6, 8k + 7 & \quad (\text{real}).
\end{aligned}$$

The Spin group  $Spin(n) \subset Cl_n$  is the subset

$$Spin(n) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^n, |x_j| = 1, l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. It is a Lie group and its Lie algebra is

$$\mathfrak{spin}(n) = \text{span}\{e_i e_j \mid 1 \leq i < j \leq n\}.$$

The restriction of  $\kappa$  to  $Spin(n)$  defines the Lie group representation

$$\kappa_n := \kappa|_{Spin(n)} : Spin(n) \longrightarrow GL(\Delta_n),$$

which is, in fact, special unitary. We have the corresponding Lie algebra representation

$$\kappa_{n*} : \mathfrak{spin}(n) \longrightarrow \mathfrak{gl}(\Delta_n).$$

Both representations can be extended to tensor powers  $\Delta_n^{\otimes m}$ ,  $m \in \mathbb{N}$ , in the usual way.

Recall that the Spin group  $Spin(n)$  is the universal double cover of  $SO(n)$ ,  $n \geq 3$ . For  $n = 2$  we consider  $Spin(2)$  to be the connected double cover of  $SO(2)$ . The covering map will be denoted by

$$\lambda_n : Spin(n) \rightarrow SO(n) \subset GL(\mathbb{R}^n).$$

Its differential is given by  $\lambda_{n*}(e_i e_j) = 2E_{ij}$ , where  $E_{ij} = e_i^* \otimes e_j - e_j^* \otimes e_i$  is the standard basis of the skew-symmetric matrices, and  $e^*$  denotes the metric dual of the vector  $e$ . Furthermore, we will abuse the notation and also denote by  $\lambda_n$  the induced representation on the exterior algebra  $\bigwedge^* \mathbb{R}^n$ .

The Clifford multiplication  $\mu_n$  is skew-symmetric with respect to the Hermitian product

$$\langle x \cdot \phi_1, \phi_2 \rangle = \langle \mu_n(x \otimes \phi_1), \phi_2 \rangle = -\langle \phi_1, \mu_n(x \otimes \phi_2) \rangle = -\langle \phi_1, x \cdot \phi_2 \rangle, \quad (1)$$

is  $Spin(n)$ -equivariant and can be extended to a  $Spin(n)$ -equivariant map

$$\begin{aligned} \mu_n : \bigwedge^* (\mathbb{R}^n) \otimes \Delta_n &\longrightarrow \Delta_n \\ \omega \otimes \psi &\mapsto \omega \cdot \psi. \end{aligned}$$

## 2.2 Spinorially twisted spin groups and representations

We define the twisted Spin group  $Spin^r(n)$  as follows

$$Spin^r(n) = (Spin(n) \times Spin(r)) / \{\pm(1, 1)\} = Spin(n) \times_{\mathbb{Z}_2} Spin(r),$$

where  $r \in \mathbb{N}$  and  $r \geq 2$ , which fit into exact sequences

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^r(n) \xrightarrow{\lambda_n \times \lambda_r} SO(n) \times SO(r) \longrightarrow 1,$$

where

$$\begin{aligned} \lambda_n \times \lambda_r : Spin^r(n) &\longrightarrow SO(n) \times SO(r) \\ [g, h] &\mapsto (\lambda_n(g), \lambda_r(h)). \end{aligned}$$

We will call  $r$  the *rank* of the twisting. Note that the groups  $Spin^2(n) = Spin^c(n)$  and  $Spin^3(n) = Spin^q(n)$ . The Lie algebra of  $Spin^r(n)$  is

$$\mathfrak{spin}^r(n) = \mathfrak{spin}(n) \oplus \mathfrak{spin}(r).$$

Consider the representations

$$\begin{aligned} \kappa_n \otimes \kappa_r^m : Spin^r(n) &\longrightarrow GL(\Delta_n \otimes \Delta_r^{\otimes m}) \\ [g, h] &\mapsto \kappa_n(g) \otimes \kappa_r^m(h), \end{aligned}$$

where  $m \in \mathbb{N}$ , which are unitary with respect to the Hermitian metric, and the map

$$\begin{aligned} \mu_n \otimes \mu_r : \left( \bigwedge^* \mathbb{R}^n \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^r \right) \otimes_{\mathbb{R}} (\Delta_n \otimes \Delta_r) &\longrightarrow \Delta_n \otimes \Delta_r \\ (w_1 \otimes w_2) \otimes (\psi \otimes \varphi) &\mapsto (w_1 \otimes w_2) \cdot (\psi \otimes \varphi) = (w_1 \cdot \psi) \otimes (w_2 \cdot \varphi). \end{aligned}$$

As in the untwisted case,  $\mu_n \otimes \mu_r$  is an equivariant homomorphism of  $Spin^r(n)$  representations. Note that we can also take tensor products with more copies of  $\Delta_r$  as follows

$$\begin{aligned} \mu_r^a := Id_{\Delta_r}^{\otimes a-1} \otimes \mu_r \otimes Id_{\Delta_r}^{\otimes m-a} : \bigwedge^* \mathbb{R}^r \otimes_{\mathbb{R}} \Delta_r^m &\longrightarrow \Delta_r^m \\ (\beta) \otimes (\varphi_1 \otimes \cdots \otimes \varphi_m) &\mapsto \varphi_1 \otimes \cdots \otimes (\mu_r(\beta \otimes \varphi_a)) \otimes \cdots \otimes \varphi_m, \end{aligned}$$

with Clifford multiplication taking place only in the  $a$ -th factor. We will also write

$$\mu_r^a(\beta \otimes \varphi_1 \otimes \cdots \otimes \varphi_m) = \mu_r^a(\beta) \cdot (\varphi_1 \otimes \cdots \otimes \varphi_m).$$

Notice that

$$\kappa_{r*}^m(f_k f_l)(\varphi_1 \otimes \cdots \otimes \varphi_m) = (\mu_r^1(f_k f_l) \cdot \varphi_1) \otimes \cdots \otimes \varphi_m + \cdots + \varphi_1 \otimes \cdots \otimes (\mu_r^m(f_k f_l) \cdot \varphi_m). \quad (2)$$

An element  $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$  will be called a *twisted spinor*, or simply a *spinor*.

**Definition 2.1** Let  $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ ,  $X, Y \in \mathbb{R}^n$  and  $(f_1, \dots, f_r)$  an orthonormal frame of  $\mathbb{R}^r$ .

- Let

$$\eta_{kl}^\phi(X, Y) = \text{Re} \langle X \wedge Y \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi, \phi \rangle$$

be the real 2-form associated to the spinor  $\phi$  where  $1 \leq k, l \leq r$ .

- Define the antisymmetric endomorphism  $\hat{\eta}_{kl}^\phi \in \text{End}^-(\mathbb{R}^n)$  by

$$X \mapsto \hat{\eta}_{kl}^\phi(X) := (X \lrcorner \eta_{kl}^\phi)^\sharp,$$

where  $X \in \mathbb{R}^n$ ,  $1 \leq k, l \leq r$ ,  $\lrcorner$  denotes contraction and  $^\sharp$  denotes metric dualization.

In fact, for any  $\xi \in \bigwedge^2(\mathbb{R}^n)^*$ , we define  $\hat{\xi} \in \text{End}^-(\mathbb{R}^n)$  by

$$\begin{aligned} \hat{\xi} : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \hat{\xi}(x) := (x \lrcorner \xi)^\sharp. \end{aligned}$$

**Lemma 2.1** Let  $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ ,  $X, Y \in \mathbb{R}^n$ ,  $(f_1 \dots, f_r)$  an orthonormal basis of  $\mathbb{R}^r$  and  $1 \leq k, l \leq r$ . Then

$$\text{Re} \langle \kappa_{r*}^m(f_k f_l) \cdot \phi, \phi \rangle = 0, \quad (3)$$

$$\text{Re} \langle X \wedge Y \cdot \phi, \phi \rangle = 0, \quad (4)$$

$$\text{Im} \langle X \wedge Y \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi, \phi \rangle = 0, \quad (5)$$

$$\text{Re} \langle X \cdot \phi, Y \cdot \phi \rangle = \langle X, Y \rangle |\phi|^2, \quad (6)$$

*Proof.* By using (1) repeatedly

$$\langle \mu_r^a(f_k f_l) \cdot \phi, \phi \rangle = -\overline{\langle \mu_r^a(f_k f_l) \phi, \phi \rangle},$$

so that (3) follows from (2).

For identity (4), recall that for  $X, Y \in \mathbb{R}^n$

$$X \wedge Y = X \cdot Y + \langle X, Y \rangle.$$

Thus

$$\langle X \wedge Y \cdot \phi, \phi \rangle = -\overline{\langle X \wedge Y \cdot \phi, \phi \rangle}.$$

Identities (5) and (6) follow similarly. □

**Remarks.**

- For  $k \neq l$ ,

$$\eta_{kl}^\phi = -\eta_{lk}^\phi.$$

- By (4),

$$\eta_{kk} \equiv 0.$$

- By (5), if  $k \neq l$ ,

$$\eta_{kl}^\phi(X, Y) = \langle X \wedge Y \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi, \phi \rangle.$$

- Note that, depending on the spinor, such 2-forms can actually be identically zero.

**Lemma 2.2** Any spinor  $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$  defines two maps (extended by linearity)

$$\begin{aligned} \bigwedge^2 \mathbb{R}^r &\longrightarrow \bigwedge^2 \mathbb{R}^n \\ f_k f_l &\longmapsto \eta_{kl}^\phi, \end{aligned}$$

and

$$\begin{aligned} \bigwedge^2 \mathbb{R}^r &\longrightarrow \text{End}(\mathbb{R}^n) \\ f_k f_l &\longmapsto \hat{\eta}_{kl}^\phi. \end{aligned}$$

□

## 2.3 Spinorially twisted spin structures on oriented Riemannian manifolds

**Definition 2.2** Let  $M$  be an oriented  $n$ -dimensional Riemannian manifold,  $P_{SO(M)}$  be its principal bundle of orthonormal frames and  $r \in \mathbb{N}$ ,  $r \geq 2$ . A  $Spin^r(n)$  structure on  $M$  consists of an auxiliary principal  $SO(r)$ -bundle  $P_{SO(r)}$  and a principal  $Spin^r(n)$ -bundle  $P_{Spin^r(n)}$  together with an equivariant  $2 : 1$  covering map

$$\Lambda : P_{Spin^r(n)} \longrightarrow P_{SO(M)} \tilde{\times} P_{SO(r)},$$

where  $\tilde{\times}$  denotes the fibered product, such that  $\Lambda(pg) = \Lambda(p)(\lambda_n \times \lambda_r)(g)$  for all  $p \in P_{Spin^r(n)}$  and  $g \in Spin^r(n)$ , where  $\lambda_n \times \lambda_r : Spin^r(n) \longrightarrow SO(n) \times SO(r)$  denotes the canonical 2-fold cover.

A manifold  $M$  admitting a  $Spin^r(n)$  structure will be called a  $Spin^r$  manifold.

**Remark.** A  $Spin^r$  manifold with trivial  $P_{SO(r)}$  auxiliary bundle is a Spin manifold. Conversely, any Spin manifold admits  $Spin^r(n)$  structures with trivial  $P_{SO(r)}$  via the inclusion  $Spin(n) \subset Spin^r(n)$  given by the elements  $[g, 1]$

**Remark.** A  $Spin^r$  manifold has the following associated vector bundles:

$$\begin{aligned} TM &= P_{Spin^r(n)} \times_{\lambda_n \times \lambda_r} (\mathbb{R}^n \times \{0\}), \\ F &= P_{Spin^r(n)} \times_{\lambda_n \times \lambda_r} (\{0\} \times \mathbb{R}^r), \\ S(TM) \otimes S(F)^{\otimes m} &= P_{Spin^r(n)} \times_{\kappa_n \otimes \kappa_r^m} (\Delta_n \otimes \Delta_r^{\otimes m}), \end{aligned}$$

where the last bundle is globally defined if  $M$  and  $m$  satisfy certain conditions. Indeed,  $S(TM) \otimes S(F)^{\otimes m}$  is defined if one of the following options holds:

- $M$  is a non-Spin  $Spin^r$  manifold and  $m$  is odd. The structure group under consideration is  $Spin^r(n)$ .
- Both  $M$  and  $F$  admit Spin structures, and  $m \in \mathbb{N}$ . The structure group under consideration is  $Spin(n) \times Spin(r)$ , so that we can consider all representations of the product group.
- $M$  is Spin,  $F$  is not Spin, and  $m$  must be even. In this case, the representation  $\Delta_r^{\otimes m}$  must factor through  $SO(r)$  in order to get a globally defined bundle. Thus, the structure group we need to consider is  $Spin(n) \times SO(r)$ . Although this case falls outside the definition of  $Spin^r$  structure, we will still consider it since one can still work with twisted spinors and twisted Dirac operators.

**Example.** Let  $r = ak + bl$ ,  $a, b \in \mathbb{N}$  and consider the real Grassmannians of oriented subspaces

$$\mathbb{G}r_k(\mathbb{R}^{k+l}) = \frac{SO(k+l)}{SO(k) \times SO(l)}.$$

There exists a homomorphism  $SO(k) \times SO(l) \rightarrow Spin^r(kl)$  providing a homogeneous  $Spin^r(kl)$ -structure on the real Grassmannian  $\mathbb{G}r_k(\mathbb{R}^{k+l})$  if

$$\begin{aligned} a &\equiv l \pmod{2}, \\ b &\equiv k \pmod{2}. \end{aligned}$$

## 2.4 Covariant derivatives on spinorially twisted Spin bundles

Let  $M$  be a  $Spin^r$   $n$ -dimensional manifold and  $F$  its auxiliary Riemannian vector bundle of rank  $r$ . Assume  $F$  is endowed with a covariant derivative  $\nabla^F$  (or equivalently, that  $P_{SO(F)}$  is endowed with a connection 1-form  $\theta$ ) and denote by  $\nabla$  the Levi-Civita covariant derivative on  $M$ . These two derivatives induce the spinor covariant derivative

$$\nabla^\theta : \Gamma(S(TM) \otimes S(F)^{\otimes m}) \longrightarrow \Gamma(T^*M \otimes S(TM) \otimes S(F)^{\otimes m})$$

$$\nabla^\theta(\psi \otimes \varphi) = d(\psi \otimes \varphi) + \left[ \frac{1}{2} \sum_{1 \leq i < j \leq n} \omega_{ji} \otimes e_i e_j \cdot \psi \right] \otimes \varphi + \psi \otimes \left[ \frac{1}{2} \sum_{1 \leq k < l \leq r} \theta_{kl} \otimes \kappa_{r*}^m(f_k f_l) \cdot \varphi \right],$$

where  $\psi \otimes \varphi \in \Gamma(S(TM) \otimes S(F)^{\otimes m})$ ,  $(e_1, \dots, e_n)$  and  $(f_1, \dots, f_r)$  are a local orthonormal frames of  $TM$  and  $F$  resp., and  $\omega_{ij}$  and  $\theta_{kl}$  are the local connection 1-forms for  $TM$  (Levi-Civita) and  $F$  respectively.

From now on, we will omit the upper and lower bounds on the indices, by declaring  $i$  and  $j$  to be the indices for the frame vectors of  $M$ , and  $k$  and  $l$  to be the indices for the local frame sections of  $F$ . Now, for any tangent vectors  $X, Y \in T_x M$ ,

$$R^\theta(X, Y)(\psi \otimes \varphi) = \left[ \frac{1}{2} \sum_{i < j} \Omega_{ij}(X, Y) e_i e_j \cdot \psi \right] \otimes \varphi + \psi \otimes \left[ \frac{1}{2} \sum_{k < l} \Theta_{kl}(X, Y) \kappa_{r*}^m(f_k f_l) \cdot \varphi \right], \quad (7)$$

where

$$\Omega_{ij}(X, Y) = \langle R^M(X, Y)(e_i), e_j \rangle \quad \text{and} \quad \Theta_{kl}(X, Y) = \langle R^F(X, Y)(f_k), f_l \rangle.$$

For  $X, Y$  vector fields and  $\phi \in \Gamma(S(TM) \otimes S(F)^{\otimes m})$  a spinor field, we also have the compatibility of the covariant derivative with Clifford multiplication,

$$\nabla_X^\theta(Y \cdot \phi) = (\nabla_X Y) \cdot \phi + Y \cdot \nabla_X^\theta \phi.$$

## 2.5 Twisted differential operators

In order to simplify notation, let  $S = S(TM) \otimes S(F)^{\otimes m}$  and  $\phi \in \Gamma(S)$ .

**Definition 2.3** *The twisted Dirac operator is the first order differential operator  $\not{D}^\theta = \not{D}^{\theta, m} : \Gamma(S) \rightarrow \Gamma(S)$  defined by*

$$\not{D}^\theta(\phi) = \sum_{i=1}^n e_i \cdot \nabla_{e_i}^\theta(\phi).$$

We will generally use the notation  $\not{D}^\theta$ , and will use the notation  $\not{D}^{\theta, m}$  whenever we want to emphasize which tensor power is involved in the twisted vector bundle being considered.

**Remark.** The twisted Dirac operator  $\not{D}^\theta$  is well-defined and formally self-adjoint on compact manifolds. Moreover, if  $h \in C^\infty(M)$ ,  $\phi \in \Gamma(S)$ , we have

$$\not{D}^\theta(h\phi) = \text{grad}(h) \cdot \phi + h \not{D}^\theta(\phi).$$

The proofs of these facts are analogous to those for the  $\text{Spin}^c$  Dirac operator [1].

**Definition 2.4** *The twisted spin connection Laplacian is the second order differential operator  $\Delta : \Gamma(S) \rightarrow \Gamma(S)$  defined as*

$$\Delta^\theta(\phi) = - \sum_{i=1}^n \nabla_{e_i}^\theta \nabla_{e_i}^\theta(\phi) - \sum_{i=1}^n \text{div}(e_i) \nabla_{e_i}^\theta(\phi).$$

## 3 Curvature identities, special spinors and twisted Dirac operator's eigenvalue estimates

Throughout this section, let  $M$  be a  $\text{Spin}^r$   $n$ -dimensional manifold,  $m \in \mathbb{N}$ , such that  $S = S(TM) \otimes S(F)^{\otimes m}$  is globally defined,  $(e_1, \dots, e_n)$  and  $(f_1, \dots, f_r)$  be local orthonormal frames of  $TM$  and  $F$  respectively.

### 3.1 Curvature calculations

**Proposition 3.1** *For  $X \in \Gamma(TM)$  and  $\phi \in \Gamma(S)$ , we have*

$$\sum_{i=1}^n e_i \cdot R^\theta(X, e_i)(\phi) = -\frac{1}{2} \text{Ric}(X) \cdot \phi + \frac{1}{2} \sum_{k < l} (X \lrcorner \Theta_{kl}) \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi. \quad (8)$$

*Proof.* By (7), if  $\phi = \psi \otimes \varphi$ ,

$$R^\theta(X, e_\alpha)(\psi \otimes \varphi) = \left[ \frac{1}{2} \sum_{i < j} \Omega_{ij}(X, e_\alpha) e_i e_j \cdot \psi \right] \otimes \varphi + \psi \otimes \left[ \frac{1}{2} \sum_{k < l} \Theta_{kl}(X, e_\alpha) \kappa_{r*}^m(f_k f_l) \cdot \varphi \right].$$

Multiply by  $e_\alpha$  and sum over  $\alpha$

$$\sum_{\alpha} e_\alpha \cdot R^\theta(X, e_\alpha)(\psi \otimes \varphi) = \left[ \frac{1}{2} \sum_{\alpha} \sum_{i < j} \Omega_{ij}(X, e_\alpha) e_\alpha e_i e_j \cdot \psi \right] \otimes \varphi + \frac{1}{2} \sum_{k < l} \left[ \sum_{\alpha} \Theta_{kl}(X, e_\alpha) e_\alpha \cdot \psi \right] \otimes \kappa_{r*}^m(f_k f_l) \cdot \varphi.$$

The term

$$\frac{1}{2} \sum_{\alpha} \sum_{i < j} \Omega_{ij}(X, e_\alpha) e_\alpha e_i e_j = -\frac{1}{2} \text{Ric}(X)$$

(cf. [1]). The second term

$$\frac{1}{2} \sum_{k < l} \left[ \sum_{\alpha} \Theta_{kl}(X, e_\alpha) e_\alpha \cdot \psi \right] \otimes \kappa_{r*}^l(f_k f_l) \cdot \varphi = \frac{1}{2} \sum_{k < l} (X \lrcorner \Theta_{kl}) \cdot \kappa_{r*}^m(f_k f_l) \cdot (\psi \otimes \varphi).$$

□

**Proposition 3.2** *Let  $\phi \in \Gamma(S)$ . Then*

$$\sum_{i,j} e_i e_j \cdot R^\theta(e_i, e_j)(\phi) = \frac{R}{2} \phi + \sum_{k < l} \Theta_{kl} \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi, \quad (9)$$

where

$$\Theta_{kl} = \sum_{i < j} \Theta_{kl}(e_i, e_j) e_i \wedge e_j.$$

*Proof.* By (8),

$$\sum_{j=1}^n e_j \cdot R^\theta(e_i, e_j)(\phi) = -\frac{1}{2} \text{Ric}(e_i) \cdot \phi + \frac{1}{2} \sum_j \sum_{k < l} \Theta_{kl}(e_i, e_j) e_j \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi,$$

By multiplying with  $e_i$  and summing over  $i$ , we get

$$\sum_{i,j} e_i e_j \cdot R^\theta(e_i, e_j)(\phi) = -\frac{1}{2} \sum_i e_i \cdot \text{Ric}(e_i) \cdot \phi + \frac{1}{2} \sum_{k < l} \left[ \sum_{i,j} \Theta_{kl}(e_i, e_j) e_i e_j \right] \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi.$$

Now,

$$-\sum_i e_i \cdot \text{Ric}(e_i) = R,$$

where  $R$  denotes the scalar curvature of  $M$ , and for  $k$  and  $l$  fixed,

$$\sum_{i,j} \Theta_{kl}(e_i, e_j) e_i e_j = 2 \sum_{i < j} \Theta_{kl}(e_i, e_j) e_i e_j = 2 \Theta_{kl}.$$

□

Now, let us denote

$$\Theta = \sum_{k < l} \Theta_{kl} \otimes f_k f_l \in \wedge^2 T^* M \otimes \wedge^2 F,$$



$$\hat{\Theta} = \sum_{k < l} \hat{\Theta}_{kl} \otimes f_k f_l \in \text{End}^-(TM) \otimes \wedge^2 F,$$

and

$$\begin{aligned} \eta^\phi &= \sum_{k < l} \eta_{kl}^\phi \otimes f_k f_l \in \wedge^2 T^*M \otimes \wedge^2 F, \\ \hat{\eta}^\phi &= \sum_{k < l} \hat{\eta}_{kl}^\phi \otimes f_k f_l \in \text{End}^-(TM) \otimes \wedge^2 F. \end{aligned}$$

In order to simplify notation, we define

$$\begin{aligned} \langle \Theta, \eta^\phi \rangle_0 &= \sum_{k < l} \sum_{i < j} \Theta_{kl}(e_i, e_j) \eta_{kl}^\phi(e_i, e_j), \\ \langle \hat{\Theta}, \hat{\eta}^\phi \rangle_1 &= \sum_{k < l} \text{tr}(\hat{\Theta}_{kl}(\hat{\eta}_{kl}^\phi)^T). \end{aligned}$$

### 3.2 Parallel spinors

**Definition 3.1** A spinor  $\phi \in \Gamma(S)$  is said to be parallel if

$$\nabla_X^\theta \phi = 0$$

for all  $X \in \Gamma(TM)$ .

**Theorem 3.1** Let  $\phi \in \Gamma(S)$  be a non-zero parallel spinor. Then

1. The spinor  $\phi$  has non-zero constant length and no zeros.
2. The Ricci tensor decomposes as follows

$$\text{Ric} = \frac{1}{|\phi|^2} \sum_{k < l} \hat{\eta}_{kl}^\phi \circ \hat{\Theta}_{kl} = \frac{1}{|\phi|^2} \sum_{k < l} \hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi.$$

3. The scalar curvature is given by

$$\text{R} = \frac{1}{|\phi|^2} \sum_{k < l} \text{tr}(\hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi) = -\frac{1}{|\phi|^2} \langle \hat{\eta}^\phi, \hat{\Theta} \rangle_1.$$

4. If the connection on the auxiliary bundle  $F$  is flat, then  $M$  is Ricci-flat.
5. If the parallel spinor  $\phi$  is such that  $\eta_{kl}^\phi = 0$  for all  $1 \leq k < l \leq r$ , then the manifold  $M$  is Ricci-flat.

*Proof.* Since the spinor  $\phi$  is parallel

$$X|\phi|^2 = \langle \nabla_X^\theta \phi, \phi \rangle + \langle \phi, \nabla_X^\theta \phi \rangle = 0.$$

Thus, a non-trivial parallel spinor has constant length and no zeros.

Since  $\phi$  is parallel, the left hand side of (8) is zero and

$$\text{Ric}(e_j) \cdot \phi = \sum_{k < l} \sum_{s=1}^n \Theta_{kl}(e_j, e_s) e_s \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi.$$

By taking the real part of the hermitian inner product with  $e_i \cdot \phi$ ,

$$\begin{aligned} \text{Re} \langle \text{Ric}(e_j) \cdot \phi, e_i \cdot \phi \rangle &= \langle \text{Ric}(e_j), e_i \rangle |\phi|^2 \\ &= |\phi|^2 \text{Ric}_{ij} \end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \text{Ric}(e_j) \cdot \phi, e_i \cdot \phi \rangle &= \sum_{k < l} \sum_{s=1}^n \Theta_{kl}(e_j, e_s) \langle e_s \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi, e_i \cdot \phi \rangle \\
&= - \sum_{k < l} \sum_s \Theta_{kl}(e_j, e_s) \eta_{kl}^\phi(e_i, e_s) \\
&= \sum_{k < l} (\hat{\eta}_{kl}^\phi \circ \hat{\Theta}_{kl})_{ij}.
\end{aligned}$$

Hence, the Ricci endomorphism satisfies

$$\begin{aligned}
|\phi|^2 \text{Ric} &= \sum_{k < l} \hat{\eta}_{kl}^\phi \circ \hat{\Theta}_{kl} \\
&= \sum_{k < l} \hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi,
\end{aligned}$$

where the last equality is due to the symmetry of Ric and the skew-symmetry of both  $\hat{\Theta}_{kl}$  and  $\hat{\eta}_{kl}^\phi$ .  $\square$

### Example of a parallel twisted spinor

Consider the subgroup

$$H := \{[g, g] \in \text{Spin}(n) \times_{\mathbb{Z}_2} \text{Spin}(n) \mid g \in \text{Spin}(n)\} \subset \text{Spin}(n) \times_{\mathbb{Z}_2} \text{Spin}(n).$$

Clearly,  $H$  is isomorphic to  $SO(n)$ , and the following diagram commutes

$$\begin{array}{ccc}
& & \text{Spin}(n) \times_{\mathbb{Z}_2} \text{Spin}(n) \\
H \cong SO(n) & \xrightarrow{\quad} & \downarrow \\
& & SO(n) \times SO(n)
\end{array}$$

**Proposition 3.3** *Every Riemannian manifold admits a spinorally twisted spin structure such that an associated spinor bundle admits a parallel spinor field.*

*Proof.* Let  $M$  be a Riemannian  $n$ -dimensional manifold. Clearly, whether or not  $M$  is Spin, it admits the twisted spin structure given by

$$\begin{array}{c}
P_{\text{Spin}^n(n)} \\
\downarrow \\
P_{SO(M)} \tilde{\times} P_{SO(M)}
\end{array},$$

where  $P_{SO(M)}$  denotes the principal bundle of orthonormal frames of  $M$ . Furthermore, by the diagram above we have a reduction of structure

$$\begin{array}{ccc}
& & P_{\text{Spin}^n(n)} \\
P_{SO(M)} & \xrightarrow{\quad} & \downarrow \\
& & P_{SO(M)} \tilde{\times} P_{SO(M)}
\end{array},$$

in such a way that the twisted spinor vector bundle  $\Delta_n \otimes \Delta_n$  can be seen as an associated vector bundle to  $P_{SO(M)}$ , which is a well known fact.

Let  $\mathcal{B}$  be the unitary basis of  $\Delta_n$  described in Section 2 and  $\gamma_n$  be the corresponding real or quaternionic structure of  $\Delta_n$ . We claim that the spinor

$$\begin{aligned}
\phi_0 &:= \sum_{\psi \in \mathcal{B}} \psi \otimes \gamma_n(\psi) \\
&= \sum_{(\varepsilon_1, \dots, \varepsilon_{[n/2]}) \in \{1, -1\}^{\times [n/2]}} C(n, \varepsilon_1, \dots, \varepsilon_{[n/2]}) u_{\varepsilon_1, \dots, \varepsilon_{[n/2]}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{[n/2]}}
\end{aligned}$$

is invariant under  $H \cong SO(n)$ , where

$$\begin{aligned} C(n; \varepsilon_1, \dots, \varepsilon_{4k}) &= (-1)^{k+\frac{1}{2} \sum_{j=1}^{2k} (\varepsilon_{2j-1}+1)} & \text{if } n = 8k, 8k+1, \\ C(n; \varepsilon_1, \dots, \varepsilon_{4k+1}) &= i(-1)^{k+\frac{1}{2} \sum_{j=1}^{2k+1} (\varepsilon_{2j-1}+1)} & \text{if } n = 8k+2, 8k+3, \\ C(n; \varepsilon_1, \dots, \varepsilon_{4k+2}) &= (-1)^{k+\frac{1}{2} \sum_{j=1}^{2k+1} (\varepsilon_{2j-1}+1)} & \text{if } n = 8k+4, 8k+5, \\ C(n; \varepsilon_1, \dots, \varepsilon_{4k+3}) &= i(-1)^{k+\frac{1}{2} \sum_{j=1}^{2k+2} (\varepsilon_{2j-1}+1)} & \text{if } n = 8k+6, 8k+7. \end{aligned}$$

We will prove the invariance by means of the Lie algebra  $Lie(H) \cong \mathfrak{so}(n)$ . Let us consider the case  $n = 8k$ . Let  $\{e_1, \dots, e_{8k}\} \subset \mathbb{R}^{8k}$  be an ordered orthonormal basis, and  $\{f_1, \dots, f_{8k}\} \subset \mathbb{R}^{8k}$  be the same ordered basis but renamed since it will refer to the auxiliary twisting bundle. Thus

$$Lie(H) = \text{span}\{e_i e_j + f_i f_j \in \mathfrak{spin}(n) \oplus \mathfrak{spin}(n) \mid 1 \leq i < j \leq n\}.$$

Let us consider one summand in  $\phi_0$ ,

$$\varphi_1 := (-1)^{\frac{1}{2} \sum_{j=1}^{2k} \varepsilon_{2j-1}} u_{\varepsilon_1, \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{4k}},$$

and focus first on

$$u_{\varepsilon_1, \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{4k}},$$

Recall that for  $1 \leq p \leq 8k$ ,

$$e_p \cdot u_{\varepsilon_1, \dots, \varepsilon_{4k}} = i^{p-2[p/2]} (-1)^{[(p+1)/2]-1} \left( \prod_{\alpha=4k-[(p+1)/2]+1+p-2[p/2]}^{4k} \varepsilon_\alpha \right) u_{\varepsilon_1, \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}}.$$

If we apply  $e_p e_q$  to it, with  $1 \leq p < q \leq n$  and  $[(q-1)/2] > [(p-1)/2]$ , we get

$$\begin{aligned} & i^{q-2[q/2]} (-1)^{[(q+1)/2]-1} \left( \prod_{\alpha=4k-[(q+1)/2]+1+q-2[q/2]}^{4k} \varepsilon_\alpha \right) \\ & \times i^{p-2[p/2]} (-1)^{[(p+1)/2]-1} \left( \prod_{\beta=4k-[(p+1)/2]+1+p-2[p/2]}^{4k} \varepsilon_\beta \right) \\ & \times u_{\varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{4k}}. \end{aligned}$$

Now, let us consider another summand in  $\phi_0$

$$\begin{aligned} \varphi_2 &:= C(8k, \varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}) \\ & \times u_{\varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, (\varepsilon_{4k-[(q+1)/2]+1}), \dots, (\varepsilon_{4k-[(p+1)/2]+1}), \dots, -\varepsilon_{4k}}, \end{aligned}$$

and focus first on

$$u_{\varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, (\varepsilon_{4k-[(q+1)/2]+1}), \dots, (\varepsilon_{4k-[(p+1)/2]+1}), \dots, -\varepsilon_{4k}}.$$

When we apply  $\kappa_{n*}^1(f_p f_q)$  to it,

$$\begin{aligned} & i^{q-2[q/2]} (-1)^{[(q+1)/2]-1} \left( \prod_{\alpha=4k-[(q+1)/2]+1+q-2[q/2]}^{4k} -\varepsilon_\alpha \right) \\ & \times i^{p-2[p/2]} (-1)^{[(p+1)/2]-1} \left( \prod_{\beta=4k-[(p+1)/2]+1+p-2[p/2]}^{4k} -\varepsilon_\beta \right) \\ & \times (-1)(-1)^{q+1}(-1)^{p+1} \\ & \times u_{\varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{4k}}. \end{aligned}$$

Now, while the coefficient of  $e_p e_q \cdot \varphi_1$  is

$$C(8k; \varepsilon_1, \dots, \varepsilon_{4k})$$

$$\begin{aligned} & \times i^{q-2[q/2]} (-1)^{[(q+1)/2]-1} \left( \prod_{\alpha=4k-[(q+1)/2]+1+q-2[q/2]}^{4k} \varepsilon_\alpha \right) \\ & \times i^{p-2[p/2]} (-1)^{[(p+1)/2]-1} \left( \prod_{\beta=4k-[(p+1)/2]+1+p-2[p/2]}^{4k} \varepsilon_\beta \right), \end{aligned}$$

the coefficient of  $\kappa_{n*}^1(f_p f_q) \cdot \varphi_2$  is

$$\begin{aligned} & C(8k, \varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}) \\ & \times (-1)^{1+[(q+1)/2]+[(p+1)/2]} \\ & \times i^{q-2[q/2]} (-1)^{[(q+1)/2]-1} \left( \prod_{\alpha=4k-[(q+1)/2]+1+q-2[q/2]}^{4k} \varepsilon_\alpha \right) \\ & \times i^{p-2[p/2]} (-1)^{[(p+1)/2]-1} \left( \prod_{\beta=4k-[(p+1)/2]+1+p-2[p/2]}^{4k} \varepsilon_\beta \right). \end{aligned}$$

By checking the possible cases in which  $[(p+1)/2]$  and  $[(q+1)/2]$  are either even or odd, these two coefficients differ by  $(-1)$ . Thus

$$e_p e_q \cdot \varphi_1 + \kappa_{n*}^1(f_p f_q) \cdot \varphi_2 = 0.$$

Clearly, every summand in  $\phi_0$  has a unique counterpart as in the previous calculation. All the other possible cases for values and parities of  $n$ ,  $p$  and  $q$  are similar. Hence  $Lie(H) \cong \mathfrak{so}(n)$  annihilates  $\phi_0$ .  $\square$

**Proposition 3.4** *The 2-forms associated to  $\phi_0$  are multiples of the elements of the stanford basis of  $\mathfrak{so}(n)$ , i.e.*

$$\eta_{pq}^{\phi_0} = 2^{[n/2]} e_p \wedge e_q.$$

*Proof.* Notice that

$$\phi_0 = \sum_{(\varepsilon_1, \dots, \varepsilon_{[n/2]}) \in \{1, -1\}^{\times [n/2]}} C(n, \varepsilon_1, \dots, \varepsilon_{[n/2]}) u_{\varepsilon_1, \dots, \varepsilon_{[n/2]}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{[n/2]}}.$$

is orthogonal to any spinor orthogonal to

$$V_0 = \text{span}\{u_{\varepsilon_1, \dots, \varepsilon_{[n/2]}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{[n/2]}} \mid (\varepsilon_1, \dots, \varepsilon_{[n/2]}) \in \{1, -1\}^{\times [n/2]}\}.$$

Thus, for  $p < q$ ,  $s < t$ ,  $(p, q) \neq (s, t)$ ,

$$\begin{aligned} \eta_{st}^{\phi_0}(e_p, e_q) &= \langle e_p e_q \cdot \kappa_{n*}^1(f_s f_t) \cdot \phi_0, \phi_0 \rangle \\ &= 0, \end{aligned}$$

since each one of the summands in  $e_p e_q \cdot \kappa_{n*}^1(f_s f_t) \cdot \phi_0$  is orthogonal to  $V_0$ .

On the other hand, if  $(p, q) = (s, t)$  with  $1 \leq p < q \leq n$ ,  $[(q-1)/2] > [(p-1)/2]$ , and

$$\varphi_1 := (-1)^{\frac{1}{2} \sum_{j=1}^{2k} \varepsilon_{2j-1}} u_{\varepsilon_1, \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, -\varepsilon_{4k}},$$

then  $e_p e_q \cdot \kappa_{n*}^1(f_p f_q) \cdot \varphi_1$  is equal to

$$\begin{aligned} & (-1)^{[(p+1)/2]+[(q+1)/2]+\frac{1}{2} \sum_{j=1}^{2k} \varepsilon_{2j-1}} \\ & \times u_{\varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, (\varepsilon_{4k-[(q+1)/2]+1}), \dots, (\varepsilon_{4k-[(p+1)/2]+1}), \dots, -\varepsilon_{4k}}. \end{aligned}$$

Clearly, this is paired with

$$\varphi_3 = C(8k; \varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k})$$

$$u_{\varepsilon_1, \dots, (-\varepsilon_{4k-[(q+1)/2]+1}), \dots, (-\varepsilon_{4k-[(p+1)/2]+1}), \dots, \varepsilon_{4k}} \otimes u_{-\varepsilon_1, \dots, (\varepsilon_{4k-[(q+1)/2]+1}), \dots, (\varepsilon_{4k-[(p+1)/2]+1}), \dots, -\varepsilon_{4k}}$$

in the hermitian product, so that

$$\langle e_p e_q \cdot \kappa_{n*}^1(f_p f_q) \cdot \varphi_1, \varphi_3 \rangle = 1,$$

for all possible cases in which  $[(p+1)/2]$  and  $[(q+1)/2]$  are either even or odd. Furthermore, all the other cases for values and parities of  $n$ ,  $p$  and  $q$  are similar. Since  $\phi_0$  is made up of  $2^{[n/2]}$  summands which satisfy the previous arguments,

$$\eta_{pq}^{\psi_0}(e_s, e_t) = 2^{[n/2]}(\delta_{ps}\delta_{qt} - \delta_{pt}\delta_{qs}).$$

□

Let us now check our curvature formulas on this example. Formula (7) becomes

$$\begin{aligned} R^\theta(X, Y)(\phi_0) &= \frac{1}{2} \sum_{1 \leq i < j \leq n} \Omega_{ij}(X, Y) e_i e_j \cdot \phi_0 + \frac{1}{2} \sum_{1 \leq k < l \leq n} \Theta_{kl}(X, Y) \kappa_{n*}^1(f_k f_l) \cdot \phi_0 \\ &= \frac{1}{2} \sum_{i < j} \langle R^M(X, Y)(e_i), e_j \rangle e_i e_j \cdot \phi_0 + \frac{1}{2} \sum_{i < j} \langle R^M(X, Y)(e_i), e_j \rangle \kappa_{n*}^1(f_i f_j) \cdot \phi_0 \\ &= \frac{1}{2} \sum_{i < j} \langle R^M(X, Y)(e_i), e_j \rangle (e_i e_j + \kappa_{n*}^1(f_i f_j)) \cdot \phi_0 \\ &= 0, \end{aligned}$$

which is consistent with the parallelness of  $\phi_0$ , and

$$\begin{aligned} \sum_{1 \leq k < l \leq n} (\hat{\Omega}_{kl} \circ \hat{\eta}_{kl}^{\phi_0})_{st} &= \sum_{1 \leq k < l \leq n} \sum_{a=1}^n (\hat{\Omega}_{kl})_{sa} (\hat{\eta}_{kl}^{\phi_0})_{at} \\ &= \sum_{a=1}^n \sum_{1 \leq k < l \leq n} \Omega_{kl}(e_s, e_a) \eta_{kl}^{\phi_0}(e_a, e_t) \\ &= 2^{[n/2]} \sum_{a=1}^n \sum_{1 \leq k < l \leq n} \langle R(e_s, e_a) e_k, e_l \rangle (\delta_{ka} \delta_{lt} - \delta_{kt} \delta_{la}) \\ &= 2^{[n/2]} \left( \sum_{a < t} \langle R(e_s, e_a) e_a, e_t \rangle - \sum_{a > t} \langle R(e_s, e_a) e_t, e_a \rangle \right) \\ &= 2^{[n/2]} \sum_a \langle R(e_a, e_s) e_t, e_a \rangle \\ &= 2^{[n/2]} \sum_a \text{Ric}_{st}. \end{aligned}$$

which is consistent with (8).

### 3.3 Killing spinors

**Definition 3.2** A spinor  $\phi \in \Gamma(S)$  is a Killing spinor if, for every  $X \in \Gamma(TM)$ ,

$$\nabla_X \phi = \mu X \cdot \phi,$$

where  $\mu \in \mathbb{C}$ .

**Proposition 3.5** Let  $\phi \in \Gamma(S)$  be a non-trivial Killing spinor.

1.  $\phi$  has no zeros.
2.  $\phi$  is an eigenspinor of the twisted Dirac operator.
3. If  $\phi$  is a real Killing spinor, i.e.  $\mu$  is real, the length of the Killing spinor  $\phi$  is constant.

4. If  $\mu$  is real, then the vector field

$$V^\phi = \sum_n \langle e_i \cdot \phi, \phi \rangle e_i$$

is a Killing vector field.

The proof is analogous to the one in the  $\text{Spin}^c$  case (cf. [1]). □

**Theorem 3.2** *Let  $\phi \in \Gamma(S)$  be a real Killing spinor. Then*

- *The Ricci tensor decomposes as follows*

$$\text{Ric} = 4(n-1)\mu^2 \text{Id}_{TM} + \frac{1}{|\phi|^2} \sum_{k < l} \hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi.$$

- *The scalar curvature is given by*

$$R = 4n(n-1)\mu^2 + \frac{1}{|\phi|^2} \sum_{k < l} \text{tr}(\hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi).$$

- *If the connection on the auxiliary bundle  $F$  is flat, then  $M$  is Einstein.*
- *If the Killing spinor  $\phi$  is such that  $\eta_{kl}^\phi = 0$  for all  $1 \leq k < l \leq r$ , then the manifold  $M$  is Einstein.*

*Proof.* The left hand side of (8) now becomes

$$\begin{aligned} \sum_{i=1}^n e_i \cdot R^\theta(e_j, e_i)(\phi) &= \sum_{i \neq j} e_i \cdot \mu^2 (e_i e_j - e_j e_i) \cdot \phi \\ &= -2\mu^2 \sum_{i \neq j} e_j \cdot \phi \\ &= -2(n-1)\mu^2 e_j \cdot \phi. \end{aligned}$$

By taking the real part of the hermitian product with  $e_t \cdot \phi$  we get

$$\begin{aligned} \text{Re} [-2(n-1)\mu^2 \langle e_j \cdot \phi, e_t \cdot \phi \rangle] &= -2(n-1)\mu^2 \langle e_j, e_t \rangle |\phi|^2 \\ &= -2|\phi|^2 (n-1)\mu^2 \delta_{jt}. \end{aligned}$$

Hence, by the calculations of the last subsection we have

$$\text{Ric} = 4(n-1)\mu^2 \text{Id}_{TM} + \frac{1}{|\phi|^2} \sum_{k < l} \hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi.$$

□

### 3.4 Generalized real Killing spinors

**Definition 3.3** *A spinor  $\phi \in \Gamma(S)$  is called a generalized Killing spinor if*

$$\nabla_X \phi = -E(X) \cdot \phi$$

*for some symmetric endomorphism  $E \in \Gamma(\text{End}(TM))$  and all  $X \in \Gamma(TM)$ .*

In this case, the left hand side of (8) is

$$\sum_{i=1}^n e_i \cdot R^\theta(e_s, e_i)(\phi) = \sum_{i=1}^n e_i \cdot R^\theta(e_s, e_i)(\phi)$$

$$\begin{aligned}
&= \sum_{i=1}^n e_i \cdot (\nabla_{e_s}^\theta \nabla_{e_i}^\theta - \nabla_{e_i}^\theta \nabla_{e_s}^\theta - \nabla_{[e_s, e_i]}^\theta) \phi \\
&= \sum_{i=1}^n e_i \cdot ((\nabla_{e_i} E)(e_s) - (\nabla_{e_s} E)(e_i) + E(e_i) \cdot E(e_s) - E(e_s) \cdot E(e_i)) \cdot \phi \\
&= \sum_{i \neq s} e_i \cdot (d^\nabla E(e_i, e_s) + E(e_i) \cdot E(e_s) - E(e_s) \cdot E(e_i)) \cdot \phi,
\end{aligned}$$

where

$$d^\nabla E(X, Y) = (\nabla_X E)(Y) - (\nabla_Y E)(X).$$

Now, if the orthonormal frame also diagonalizes  $E$ , for  $i \neq s$ ,

$$\begin{aligned}
E(e_i) \cdot E(e_s) - E(e_s) \cdot E(e_i) &= E_{ii} E_{ss} (e_i \cdot e_s - e_s \cdot e_i) \\
&= 2E_{ii} E_{ss} e_i \cdot e_s,
\end{aligned}$$

and

$$e_i \cdot (E(e_i) \cdot E(e_s) - E(e_s) \cdot E(e_i)) = -2E_{ii} E_{ss} e_s,$$

so that

$$\begin{aligned}
\sum_{i \neq s} e_i \cdot (E(e_i) \cdot E(e_s) - E(e_s) \cdot E(e_i)) &= -2 \left( \sum_{i \neq s} E_{ii} \right) E_{ss} e_s \\
&= -2 \mathbf{tr}(E) E_{ss} e_s + 2E_{ss}^2 e_s \\
&= 2E_{ss} (E_{ss} - \mathbf{tr}(E)) e_s.
\end{aligned}$$

By taking the real part of the hermitian product with  $e_t \cdot \phi$

$$\begin{aligned}
\operatorname{Re} \left\langle \sum_{i \neq s} e_i \cdot (E(e_i) \cdot E(e_s) - E(e_s) \cdot E(e_i)) \cdot \phi, e_t \cdot \phi \right\rangle &= \operatorname{Re} \langle 2E_{ss} (E_{ss} - \mathbf{tr}(E)) e_s \cdot \phi, e_t \cdot \phi \rangle \\
&= 2E_{ss} (E_{ss} - \mathbf{tr}(E)) \operatorname{Re} \langle e_s \cdot \phi, e_t \cdot \phi \rangle \\
&= 2E_{ss} (E_{ss} - \mathbf{tr}(E)) \delta_{ts} |\phi|^2,
\end{aligned}$$

which gives the matrix

$$2|\phi|^2 (E^2 - \mathbf{tr}(E)E).$$

On the other hand,

$$\begin{aligned}
\sum_{i \neq s} e_i \cdot d^\nabla E(e_i, e_s) &= \sum_{i \neq s} (e_i \wedge d^\nabla E(e_i, e_s) - \langle e_i, d^\nabla E(e_i, e_s) \rangle) \\
&= \sum_{i \neq s} \left( e_i \wedge \left( \sum_{j=1}^n \langle d^\nabla E(e_i, e_s), e_j \rangle e_j \right) - \langle e_i, d^\nabla E(e_i, e_s) \rangle \right) \\
&= \sum_{i \neq s} \left( \sum_{j=1}^n \langle d^\nabla E(e_i, e_s), e_j \rangle e_i \wedge e_j - \langle e_i, d^\nabla E(e_i, e_s) \rangle \right) \\
&= \sum_{i \neq s} \left( \sum_{j \neq i} \langle d^\nabla E(e_i, e_s), e_j \rangle e_i \cdot e_j - \langle e_i, d^\nabla E(e_i, e_s) \rangle \right)
\end{aligned}$$

By taking the real part of the hermitian product with  $e_t \cdot \phi$  we get

$$\operatorname{Re} \left\langle \sum_{i \neq s} e_i \cdot d^\nabla E(e_i, e_s) \cdot \phi, e_t \cdot \phi \right\rangle = \operatorname{Re} \left\langle \sum_{i \neq s} \left( \sum_{j \neq i} \langle d^\nabla E(e_i, e_s), e_j \rangle e_i \cdot e_j - \langle e_i, d^\nabla E(e_i, e_s) \rangle \right) \cdot \phi, e_t \cdot \phi \right\rangle$$

$$\begin{aligned}
&= \operatorname{Re} \left\langle \sum_{i \neq s} \left( \sum_{j \neq i} \langle d^\nabla E(e_i, e_s), e_j \rangle e_i \cdot e_j \right) \cdot \phi, e_t \cdot \phi \right\rangle \\
&= - \sum_{i \neq s} \sum_{j \neq i} \langle d^\nabla E(e_i, e_s), e_j \rangle \operatorname{Re} \langle e_t \cdot e_i \cdot e_j \cdot \phi, \phi \rangle \\
&= \sum_{i,j} \langle d^\nabla E(e_s, e_i), e_j \rangle \nu^\phi(e_t, e_i, e_j) \\
&= \sum_{i,j} \langle (e_s \lrcorner d^\nabla E)(e_i), e_j \rangle (e_t \lrcorner \nu^\phi)(e_i, e_j) \\
&= \sum_{i,j} (e_s \lrcorner d^\nabla E)_{ji} (e_t \lrcorner \nu^\phi)_{ji} \\
&=: ((\lrcorner d^\nabla E) \circledast (\lrcorner \nu^\phi))_{ts},
\end{aligned} \tag{10}$$

where

$$\nu^\phi(e_t, e_i, e_j) := \operatorname{Re} \langle e_t \cdot e_i \cdot e_j \cdot \phi, \phi \rangle. \tag{11}$$

Thus,

$$\operatorname{Ric}_{st} = -2E_{ss}(E_{ss} - \operatorname{tr}(E))\delta_{st} - (\lrcorner d^\nabla E \circledast \lrcorner \nu^\phi)_{st} + \sum_{k < l} (\hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi)_{st},$$

i.e.

$$\operatorname{Ric} = -2E^2 + 2\operatorname{tr}(E)E - (\lrcorner d^\nabla E \circledast \lrcorner \nu^\phi) + \sum_{k < l} \hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi.$$

and

$$R = -2\operatorname{tr}(E^2) + 2\operatorname{tr}(E)^2 - \operatorname{tr}(\lrcorner d^\nabla E \circledast \lrcorner \nu^\phi) + \sum_{k < l} \operatorname{tr}(\hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi).$$

Thus, we have proved the following.

**Theorem 3.3** *Let  $\phi \in \Gamma(S)$  be a generalized Killing spinor. Then*

- *the Ricci tensor decomposes as follows*

$$\operatorname{Ric} = -2E^2 + 2\operatorname{tr}(E)E - (\lrcorner d^\nabla E \circledast \lrcorner \nu^\phi) + \sum_{k < l} \hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi,$$

where  $(\lrcorner d^\nabla E \circledast \lrcorner \nu^\phi)$  and  $\nu^\phi$  are defined as in (10) and (11) respectively;

- *the scalar curvature is given by*

$$R = -2\operatorname{tr}(E^2) + 2\operatorname{tr}(E)^2 - \operatorname{tr}(\lrcorner d^\nabla E \circledast \lrcorner \nu^\phi) + \sum_{k < l} \operatorname{tr}(\hat{\Theta}_{kl} \circ \hat{\eta}_{kl}^\phi).$$

□

**Remark.** These formulas reduce to the previous two cases when  $E$  is a multiple of the identity endomorphism  $E = \mu \operatorname{Id}_{TM}$ .

### 3.5 Twisted Schrödinger-Lichnerowicz formula

Recall the curvature operator

$$\Theta = \sum_{k < l} \Theta_{kl} \otimes f_k f_l \in \wedge^2 TM \otimes \wedge^2 F$$

of the connection  $F$ , and denote by

$$\tilde{\Theta}^m = (\mu_n \otimes \kappa_{r*}^m)(\Theta)$$



the corresponding operator on twisted spinor fields. For future use, note the following operator norm inequality

$$|\tilde{\Theta}^m| \leq m|\tilde{\Theta}^1|,$$

which follows from (2).

**Theorem 3.4 (Twisted Schrödinger-Lichnerowicz Formula)** *Let  $\phi \in \Gamma(S)$ . Then*

$$\tilde{\partial}^\theta(\tilde{\partial}^\theta(\phi)) = \Delta^\theta(\phi) + \frac{R}{4}\phi + \frac{1}{2}\tilde{\Theta}^m \cdot \phi \quad (12)$$

where  $R$  is the scalar curvature of the Riemannian manifold  $M$ .

*Proof.* Consider the difference

$$\begin{aligned} \tilde{\partial}^\theta(\tilde{\partial}^\theta(\phi)) - \Delta^\theta(\phi) &= \sum_{i,j} e_i \cdot \nabla_{e_i}^\theta (e_j \cdot \nabla_{e_j}^\theta \phi) + \sum_i \nabla_{e_i}^\theta \nabla_{e_i}^\theta \phi + \sum_i \operatorname{div}(e_i) \nabla_{e_i}^\theta \phi \\ &= \sum_{i,j} e_i \cdot (\nabla_{e_i} e_j \cdot \nabla_{e_i}^\theta \phi + e_j \cdot \nabla_{e_j}^\theta \nabla_{e_i}^\theta \phi) + \sum_i \nabla_{e_i}^\theta \nabla_{e_i}^\theta \phi + \sum_i \operatorname{div}(e_i) \nabla_{e_i}^\theta \phi \\ &= \sum_{i,j,k} \langle \nabla_{e_i} e_j, e_k \rangle e_i e_k \cdot \nabla_{e_i}^\theta \phi + \sum_{i,j} e_i e_j \cdot \nabla_{e_j}^\theta \nabla_{e_i}^\theta \phi \\ &\quad + \sum_i \nabla_{e_i}^\theta \nabla_{e_i}^\theta \phi + \sum_i \operatorname{div}(e_i) \nabla_{e_i}^\theta \phi \\ &= \sum_i \sum_{j \neq k} \langle \nabla_{e_i} e_j, e_k \rangle e_i e_k \cdot \nabla_{e_i}^\theta \phi + \sum_{i \neq j} e_i e_j \cdot \nabla_{e_j}^\theta \nabla_{e_i}^\theta \phi, \end{aligned}$$

since

$$\sum_j \sum_{i=k} \langle \nabla_{e_i} e_j, e_k \rangle e_i e_k \cdot \nabla_{e_j}^\theta \phi = - \sum_j \operatorname{div}(e_j) \nabla_{e_j}^\theta \phi.$$

Now, for fixed  $j$

$$\sum_{i \neq k} \langle \nabla_{e_i} e_j, e_k \rangle e_i e_k = \sum_{i < k} \langle e_j, [e_k, e_i] \rangle e_i e_k.$$

Thus,

$$\begin{aligned} \tilde{\partial}^\theta(\tilde{\partial}^\theta(\phi)) - \Delta^\theta(\phi) &= \sum_j \sum_{i < k} \langle e_j, [e_k, e_i] \rangle e_i e_k \cdot \nabla_{e_i}^\theta \phi + \sum_{i < j} e_i e_j \cdot (\nabla_{e_i}^\theta \nabla_{e_j}^\theta - \nabla_{e_j}^\theta \nabla_{e_i}^\theta) \phi \\ &= \sum_{i < j} e_i e_j \cdot (\nabla_{e_i}^\theta \nabla_{e_j}^\theta - \nabla_{e_j}^\theta \nabla_{e_i}^\theta - \nabla_{[e_i, e_j]}^\theta) \phi \\ &= \frac{1}{2} \sum_{i,j} e_i e_j R^\theta(e_i, e_j) \phi. \end{aligned}$$

The result follows from Proposition 9. □

### 3.6 Bochner-type arguments

In this subsection we will prove some corollaries of the Schrödinger-Lichnerowicz formula and Bochner type arguments as in [1, 3, 8]. From here onwards, we will assume that the  $n$ -dimensional Riemannian Spin<sup>r</sup> manifold  $M$  is compact (without border) and connected.

#### 3.6.1 Harmonic spinors

**Corollary 3.1** *If  $R \geq 2m|\tilde{\Theta}^1|$  everywhere (in point-wise operator norm), and the inequality is strict at a point, then*

$$\ker(\tilde{\partial}^{\theta,m}) = 0.$$

Furthermore,

$$\ker(\tilde{\mathcal{D}}^{\theta, m'}) = 0$$

for any  $0 \leq m' \leq m$  such that the bundle  $S(TM) \otimes S(F)^{\otimes m'}$  is globally defined.

*Proof.* If  $\phi \neq 0$  is a solution of

$$\tilde{\mathcal{D}}^{\theta}(\phi) = 0,$$

by the twisted Schrödinger-Lichnerowicz formula (12)

$$0 = \Delta_{\theta}(\phi) + \frac{R}{4}\phi + \frac{1}{2}\tilde{\Theta}^m \cdot \phi.$$

By taking hermitian product with  $\phi$  and integrating over  $M$  we get

$$\begin{aligned} 0 &= \int_M \langle \Delta_{\theta}(\phi), \phi \rangle + \int_M \frac{R}{4} \langle \phi, \phi \rangle + \frac{1}{2} \int_M \langle \tilde{\Theta}^m \cdot \phi, \phi \rangle \\ &\geq \int_M |\nabla^{\theta} \phi|^2 + \frac{1}{4} \int_M (R - 2|\tilde{\Theta}^m|) |\phi|^2 \\ &\geq \int_M |\nabla^{\theta} \phi|^2 + \frac{1}{4} \int_M (R - 2m|\tilde{\Theta}^1|) |\phi|^2. \end{aligned}$$

Since

$$R - 2m|\tilde{\Theta}^1| \geq 0,$$

then

$$|\nabla^{\theta} \phi| = 0,$$

so that  $\phi$  is parallel, has non-zero constant length and no zeros. Furthermore, since

$$R - 2m|\tilde{\Theta}^1| > 0$$

at some point,

$$0 \geq |\phi|^2 \int_M (R - 2m|\tilde{\Theta}^1|) > 0,$$

which is a contradiction.

The last claim now follows from

$$R \geq R - 2|\tilde{\Theta}^1| \geq R - 4|\tilde{\Theta}^1| \geq \dots \geq R - 2m|\tilde{\Theta}^1| \geq 0.$$

□

### Remarks.

- The last statement of Corollary 3.1 means that if there are no harmonic spinors for a given power due to the condition on the curvatures  $R$  and  $\Theta^1$ , then there are no harmonic spinors for the twisted spinor bundles with smaller powers of  $\Delta_r$  either.
- One can prove that a compact Riemannian  $n$ -dimensional manifold carrying a non-flat parallel even Clifford structure of rank  $r$  (cf. [6]) is a  $\text{Spin}^r$  manifold and carries no harmonic spinors for

$$m \leq \frac{n + 8r - 16}{r(r - 1)} \quad (13)$$

if the scalar curvature is non-negative.

In particular, the case of rank  $r = 3$  corresponds to quaternion-Kähler manifolds, and Corollary 3.1 and (13) reproduce some of the vanishings of indices of twisted Dirac operators proved (via twistor transform) in [9].

Now notice that

$$\begin{aligned}
\langle \tilde{\Theta}^m \cdot \phi, \phi \rangle &= \left\langle \sum_{k < l} \left[ \sum_{i < j} \Theta_{kl}(e_i, e_j) e_i e_j \right] \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi, \phi \right\rangle \\
&= \sum_{k < l} \sum_{i < j} \Theta_{kl}(e_i, e_j) \eta_{kl}^\phi(e_i, e_j) \\
&= \langle \Theta, \eta^\phi \rangle_0,
\end{aligned} \tag{14}$$

which is a real number dependent on the curvature of the connection on  $F$  and on the specific spinor  $\phi$ .

**Corollary 3.2** *If  $\phi$  is such that*

$$R|\phi|^2 + 2 \langle \Theta, \eta^\phi \rangle_0 \geq 0$$

*everywhere, and the inequality is strict at a point, then*

$$\not\partial^\theta(\phi) \neq 0.$$

*Proof.* Suppose  $\phi \neq 0$  is such that

$$\not\partial^\theta(\phi) = 0.$$

Then, by (12)

$$\begin{aligned}
0 &= \int_M \langle \Delta^\theta(\phi), \phi \rangle + \int_M \frac{R}{4} \langle \phi, \phi \rangle + \frac{1}{2} \int_M \langle \tilde{\Theta} \cdot \phi, \phi \rangle \\
&= \int_M |\nabla^\theta \phi|^2 + \frac{1}{4} \int_M (R|\phi|^2 + 2 \langle \Theta, \eta^\phi \rangle_0) \\
&\geq 0,
\end{aligned}$$

so that  $\phi$  is parallel, has non-zero constant length and no zeros. Since

$$R|\phi|^2 + 2 \langle \Theta, \eta^\phi \rangle_0 > 0$$

at some point,

$$0 \geq \int_M (R|\phi|^2 + 2 \langle \Theta, \eta^\phi \rangle_0) > 0$$

which is a contradiction.  $\square$

**Remark.** This corollary means that one can check that a spinor is not harmonic by using the scalar curvature of the manifold, the curvature operator of the connection on  $F$  and the 2-forms associated to the spinor.

### 3.6.2 Killing spinors

**Corollary 3.3** *Suppose  $\phi \neq 0$  is a Killing spinor with Killing constant  $\mu$ . Then  $\mu$  is either real or imaginary, and*

$$\mu^2 \geq \frac{1}{4n^2} \min_M (R - 2|\tilde{\Theta}^m|) \geq \frac{1}{4n^2} \min_M (R - 2m|\tilde{\Theta}^1|).$$

*If either of the two inequalities is attained, then  $\phi$  is parallel, i.e.  $\mu = 0$ .*

*Proof.* Recall that

$$\begin{aligned}
\not\partial^\theta(\phi) &= \sum_{i=1}^n e_i \cdot \nabla_{e_i}^\theta \phi \\
&= -n\mu\phi.
\end{aligned}$$

Then, by the twisted Schrödinger-Lichnerowicz formula (12)

$$n^2\mu^2\phi = \Delta^\theta(\phi) + \frac{R}{4}\phi + \frac{1}{2}\tilde{\Theta}^m \cdot \phi.$$

By taking hermitian product with  $\phi$  and integrating over  $M$  we get

$$\begin{aligned} n^2\mu^2 \int_M |\phi|^2 &= \int_M |\nabla^\theta \phi|^2 + \int_M \frac{R}{4} |\phi|^2 + \int_M \frac{1}{2} \langle \tilde{\Theta}^m \cdot \phi, \phi \rangle \\ &\geq \frac{1}{4} \int_M (R - 2|\tilde{\Theta}^m|) |\phi|^2 \\ &\geq \frac{1}{4} \min_M (R - 2|\tilde{\Theta}^m|) \int_M |\phi|^2 \\ &\geq \frac{1}{4} \min_M (R - 2m|\tilde{\Theta}^1|) \int_M |\phi|^2, \end{aligned}$$

and the inequalities follow. Since the right hand side of the equality above is a real number,  $\mu$  must be either real or imaginary.

Now, if either of the inequalities is attained,

$$\int_M |\nabla^\theta \phi|^2 = 0 \quad \text{and} \quad \nabla^\theta \phi = 0.$$

□

**Corollary 3.4** *Suppose  $\phi \neq 0$  is a real Killing spinor with Killing constant  $\mu$ . Then,*

$$\mu^2 \geq \frac{1}{4n^2 \text{vol}(M)} \int_M \left[ R + \frac{2}{|\phi|^2} \langle \Theta, \eta^\phi \rangle_0 \right],$$

and

$$\mu^2 \geq \frac{1}{4n|\phi|^2 \text{vol}(M)} \int_M 2 \langle \Theta, \eta^\phi \rangle_0 - \langle \hat{\Theta}, \hat{\eta}^\phi \rangle_1.$$

*Proof.* By the twisted Schrödinger-Lichnerowicz formula (12)

$$n^2\mu^2\phi = \Delta^\theta(\phi) + \frac{R}{4}\phi + \frac{1}{2}\tilde{\Theta}^m \cdot \phi.$$

By taking hermitian product with  $\phi$  and integrating we get

$$\begin{aligned} n^2\mu^2 \int_M |\phi|^2 &= \int_M \langle \Delta^\theta(\phi), \phi \rangle + \int_M \frac{R}{4} \langle \phi, \phi \rangle + \frac{1}{2} \int_M \langle \tilde{\Theta}^m \cdot \phi, \phi \rangle, \\ &\geq \frac{1}{4} \int_M R |\phi|^2 + 2 \langle \Theta, \eta^\phi \rangle_0. \end{aligned}$$

Since  $|\phi|$  is a non-zero constant

$$\begin{aligned} \mu^2 &\geq \frac{1}{4n^2|\phi|^2 \text{vol}(M)} \int_M [R|\phi|^2 + 2 \langle \Theta, \eta^\phi \rangle_0] \\ &= \frac{1}{4n^2|\phi|^2 \text{vol}(M)} \int_M [4n(n-1)\mu^2|\phi|^2 - \langle \hat{\Theta}, \hat{\eta}^\phi \rangle_1 + 2 \langle \Theta, \eta^\phi \rangle_0] \\ &= \frac{(n-1)\mu^2}{n} + \frac{1}{4n^2|\phi|^2 \text{vol}(M)} \int_M 2 \langle \Theta, \eta^\phi \rangle_0 - \langle \hat{\Theta}, \hat{\eta}^\phi \rangle_1, \end{aligned}$$

where we have used Theorem 3.2.

□

### 3.6.3 Dirac eigen-spinors

**Corollary 3.5** *Suppose  $\phi$  is a Dirac eigenspinor*

$$\not\partial^\theta \phi = \lambda \phi.$$

for some  $\lambda \in \mathbb{R}$ . Then

$$\lambda^2 \geq \frac{n}{4(n-1)} \left( \min_M (\mathbf{R} - 2|\tilde{\Theta}^m|) \right) \geq \frac{n}{4(n-1)} \left( \min_M (\mathbf{R} - 2m|\tilde{\Theta}^1|) \right).$$

If either of the lower bounds is non-negative and is attained, the spinor  $\phi$  is a real Killing spinor with Killing constant

$$\mu = \pm \frac{1}{2} \sqrt{\frac{1}{n(n-1)} \min_M (\mathbf{R} - 2|\tilde{\Theta}^m|)} \quad \text{or} \quad \mu = \pm \frac{1}{2} \sqrt{\frac{1}{n(n-1)} \min_M (\mathbf{R} - 2m|\tilde{\Theta}^1|)},$$

respectively.

*Proof.* Following [1], let  $h : M \rightarrow \mathbb{R}$  be a fixed smooth function. Consider the following metric connection on the twisted spin bundle

$$\nabla_X^h \phi = \nabla_X^\theta \phi + hX \cdot \phi.$$

Let

$$\Delta^h(\phi) = - \sum_{i=1}^n \nabla_{e_i}^h \nabla_{e_i}^h \phi - \sum_{i=1}^n \operatorname{div}(e_i) \nabla_{e_i}^h \phi,$$

be the Laplacian for this connection and recall that

$$|\nabla^h \phi|^2 = \sum_{i=1}^n |\nabla_{e_i}^\theta \phi + h e_i \cdot \phi|^2.$$

Then, by (12)

$$\begin{aligned} (\not\partial^\theta - h) \circ (\not\partial^\theta - h)(\phi) &= \not\partial^\theta(\not\partial^\theta \phi) - 2h\not\partial^\theta \phi - \operatorname{grad}(h) \cdot \phi + h^2 \phi \\ &= \Delta^\theta(\phi) + \frac{\mathbf{R}}{4} \phi + \frac{1}{2} \tilde{\Theta}^m \cdot \phi - 2h\not\partial^\theta \phi - \operatorname{grad}(h) \cdot \phi + h^2 \phi. \end{aligned}$$

On the other hand,

$$\Delta^h \phi = \Delta^\theta \phi - 2h\not\partial^\theta \phi - \operatorname{grad}(h) \cdot \phi + h^2 \phi.$$

Thus

$$(\not\partial^\theta - h) \circ (\not\partial^\theta - h)(\phi) = \Delta^h(\phi) + \frac{\mathbf{R}}{4} \phi + \frac{1}{2} \tilde{\Theta}^m \cdot \phi + (1-n)h^2 \phi$$

By using  $\not\partial^\theta \phi = \lambda \phi$ , setting  $h = \frac{\lambda}{n}$ , taking hermitian product with  $\phi$  and integrating over  $M$  we get

$$\lambda^2 \left( \frac{n-1}{n} \right)^2 \int_M |\phi|^2 = \int_M |\nabla^{\lambda/n} \phi|^2 + \lambda^2 \frac{1-n}{n^2} \int_M |\phi|^2 + \int_M \frac{\mathbf{R}}{4} |\phi|^2 + \int_M \frac{1}{2} \langle \tilde{\Theta}^m \cdot \phi, \phi \rangle$$

so that

$$\begin{aligned} \lambda^2 \left( \frac{n-1}{n} \right) \int_M |\phi|^2 &= \int_M |\nabla^{\lambda/n} \phi|^2 + \int_M \frac{\mathbf{R}}{4} |\phi|^2 + \int_M \frac{1}{2} \langle \tilde{\Theta}^m \cdot \phi, \phi \rangle \\ &\geq \frac{1}{4} \min_M (\mathbf{R} - 2|\tilde{\Theta}^m|) \int_M |\phi|^2 \\ &\geq \frac{1}{4} \min_M (\mathbf{R} - 2m|\tilde{\Theta}^1|) \int_M |\phi|^2, \end{aligned}$$

and

$$\lambda^2 \geq \frac{n}{4(n-1)} \min_M (\mathbf{R} - 2|\tilde{\Theta}^m|) \geq \frac{n}{4(n-1)} \min_M (\mathbf{R} - 2m|\tilde{\Theta}^1|).$$

If either of the lower bounds is attained,

$$\int_M |\nabla^{\lambda/n} \phi|^2 = 0,$$

i.e.

$$\nabla^{\lambda/n} \phi = 0.$$

□

Now, let  $E \in \text{End}(TM)$  be a fixed symmetric endomorphism and consider the following metric connection on the twisted spin bundle

$$\nabla_X^E \phi = \nabla_X \phi + E(X) \cdot \phi.$$

Let

$$\Delta^E(\phi) = - \sum_{i=1}^n \nabla_{e_i}^E \nabla_{e_i}^E(\phi) - \sum_{i=1}^n \text{div}(e_i) \nabla_{e_i}^E(\phi),$$

be this connection's Laplacian and

$$\begin{aligned} |\nabla^E \phi|^2 &= \sum_{i=1}^n |\nabla_{e_i} \phi + E(e_i) \cdot \phi|^2 \\ &= \sum_{i=1}^n |\nabla_{e_i} \phi|^2 - 2\text{Re} \langle E(e_i) \cdot \nabla_{e_i} \phi, \phi \rangle + |E(e_i)|^2 |\phi|^2 \\ &= |\nabla \phi|^2 + |E|^2 |\phi|^2 - 2\text{Re} \sum_{i=1}^n \langle E(e_i) \cdot \nabla_{e_i} \phi, \phi \rangle. \end{aligned} \tag{15}$$

On the complement of the zero set of a spinor  $\phi$ , we can define a symmetric bilinear form  $Q_\phi$  by

$$Q_\phi(X, Y) = \frac{1}{2} \text{Re} \left\langle X \cdot \nabla_Y^\theta \phi + Y \cdot \nabla_X^\theta \phi, \frac{\phi}{|\phi|^2} \right\rangle$$

The associated field of quadratic forms gives

$$Q_\phi(e_i) = \text{Re} \left\langle e_i \cdot \nabla_{e_i}^\theta \phi, \frac{\phi}{|\phi|^2} \right\rangle$$

and

$$\text{tr}(Q_\phi) = \text{Re} \left\langle \not{\partial}^\theta \phi, \frac{\phi}{|\phi|^2} \right\rangle$$

If  $\phi \neq 0$  is a Dirac spinor  $\not{\partial}^\theta(\phi) = \lambda \phi$ , then

$$\text{tr}(Q_\phi) = \lambda.$$

So, let us take

$$\begin{aligned} E(X) &= (X \lrcorner Q_\phi)^\sharp \\ &= \sum_{i=1}^n Q_\phi(X, e_i) e_i \\ &= \ell^\phi(X), \end{aligned}$$

the so-called *energy-momentum tensor* of  $\phi$ . Then, we can examine further the second and third summands of the identity (15) which now looks as follows

$$|\nabla^{\ell^\phi} \phi|^2 = |\nabla \phi|^2 + |\ell^\phi|^2 |\phi|^2 - 2\text{Re} \sum_{i=1}^n \langle \ell^\phi(e_i) \cdot \nabla_{e_i} \phi, \phi \rangle.$$

On the one hand,

$$|\ell^\phi|^2 = \sum_{i=1}^n |\ell^\phi(e_i)|^2 = \sum_{i,j=1}^n Q_\phi(e_i, e_j)^2,$$

and on the other,

$$\begin{aligned}
-2\operatorname{Re} \sum_{i=1}^n \langle \ell^\phi(e_i) \cdot \nabla_{e_i} \phi, \phi \rangle &= -2\operatorname{Re} \sum_{i=1}^n \left\langle \left( \sum_{j=1}^n Q_\phi(e_i, e_j) e_j \right) \cdot \nabla_{e_i} \phi, \phi \right\rangle \\
&= -2|\phi|^2 \sum_{i,j=1}^n Q_\phi(e_i, e_j)^2.
\end{aligned}$$

Thus,

$$|\nabla \phi|^2 = |\nabla^{\ell^\phi} \phi|^2 + |\ell^\phi|^2 |\phi|^2,$$

so that

$$\int_M |\not\partial^\theta \phi|^2 = \int_M |\nabla^{\ell^\phi} \phi|^2 + |\ell^\phi|^2 |\phi|^2 + \frac{R}{4} |\phi|^2 + \frac{1}{2} \langle \tilde{\Theta}^m \cdot \phi, \phi \rangle.$$

Since  $\phi$  is a Dirac eigenspinor with eigenvalue  $\lambda$ ,

$$\begin{aligned}
\lambda^2 \int_M |\phi|^2 &\geq \int_M |\ell^\phi|^2 |\phi|^2 + \frac{R}{4} |\phi|^2 + \frac{1}{2} \langle \tilde{\Theta}^m \cdot \phi, \phi \rangle \\
&\geq \min_M \left( |\ell^\phi|^2 + \frac{R}{4} - \frac{1}{2} |\tilde{\Theta}^m| \right) \int_M |\phi|^2 \\
&\geq \min_M \left( |\ell^\phi|^2 + \frac{R}{4} - \frac{m}{2} |\tilde{\Theta}^1| \right) \int_M |\phi|^2.
\end{aligned}$$

i.e.

$$\lambda^2 \geq \min_M \left( |\ell^\phi|^2 + \frac{R}{4} - \frac{1}{2} |\tilde{\Theta}^m| \right) \geq \min_M \left( |\ell^\phi|^2 + \frac{R}{4} - \frac{m}{2} |\tilde{\Theta}^1| \right).$$

If the lower bound is attained,

$$\int_M |\nabla^{\ell^\phi} \phi|^2 = 0 \quad \text{and} \quad \nabla^{\ell^\phi} \phi = 0,$$

i.e.

$$\nabla_X \phi = -\ell^\phi(X) \cdot \phi$$

for all  $X \in \Gamma(TM)$ . Furthermore, since  $\nabla^{\ell^\phi}$  is compatible with the metric,  $|\phi|$  is constant.

Thus, we have proved the following.

**Corollary 3.6** *Suppose  $\phi \neq 0$  is a Dirac eigenspinor*

$$\not\partial^\theta \phi = \lambda \phi.$$

*Then*

$$\lambda^2 \geq \min_M \left( |\ell^\phi|^2 + \frac{R}{4} - \frac{1}{2} |\tilde{\Theta}^m| \right) \geq \min_M \left( |\ell^\phi|^2 + \frac{R}{4} - \frac{m}{2} |\tilde{\Theta}^1| \right),$$

*where  $\ell^\phi$  is the energy-momentum tensor of  $\phi$ . If either of the lower bounds is non-negative and is attained,  $\phi$  has constant length and no zeros, and is a generalized Killing spinor with symmetric endomorphism  $\ell^\phi$ , i.e.*

$$\nabla_X \phi = -\ell^\phi(X) \cdot \phi$$

*for all  $X \in \Gamma(TM)$ .*

□

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